Scalar Quantum Field Theory with Cubic Interaction

Carl M. Bender*, Dorje C. Brody, and Hugh F. Jones

Blackett Laboratory, Imperial College, London SW7 2BZ, UK

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In this paper it is shown that an $i\varphi^3$ field theory *is* a physically acceptable field theory model (the spectrum is positive and the theory is unitary). The demonstration rests on the perturbative construction of a linear operator $\mathcal{C}$, which is needed to define the Hilbert space inner product. The $\mathcal{C}$ operator is a new, time-independent observable in $\mathcal{PT}$-symmetric quantum field theory.

A scalar $g\varphi^3$ field theory is often used as a pedagogical example of perturbative renormalization even though this model is not physically realistic (the energy is not bounded below). However, we argue that when $g = i\epsilon$ is imaginary, one obtains a fully acceptable quantum field theory and we show how to construct perturbatively the Hilbert space in which cubic scalar field theories in $(D+1)$-dimensional Minkowski space-time are *self-adjoint*. Consequently, such theories have positive spectra and exhibit unitary time evolution. We consider here such complex field-theoretic Hamiltonians as

$$H = \int d\mathbf{x} \left[\tfrac{1}{2}\pi_{\mathbf{x}}^2 + \tfrac{1}{2}(\nabla\varphi_{\mathbf{x}})^2 + \tfrac{1}{2}\mu^2\varphi_{\mathbf{x}}^2 + i\epsilon\varphi_{\mathbf{x}}^3\right], \qquad (1)$$

where $\int d\mathbf{x} = \int d^D x$. [We suppress the time variable t in the fields and write $\varphi(\mathbf{x}, t) = \varphi_{\mathbf{x}}$.] The fields in (1) satisfy the ETCR $[\varphi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y})$. As in quantum mechanics, where the operators x and p change sign under parity reflection $\mathcal{P}$, we assume that the fields are *pseudoscalars* and also change sign under $\mathcal{P}$: $\mathcal{P}\varphi(\mathbf{x}, t)\mathcal{P} = -\varphi(-\mathbf{x}, t)$ and $\mathcal{P}\pi(\mathbf{x}, t)\mathcal{P} = -\pi(-\mathbf{x}, t)$. Field-theory models like that in (1) are of physical interest because they arise in Reggeon field theory and in the study of the Lee-Yang edge singularity [1].

This paper is motivated by the observation that the cubic complex quantum-mechanical Hamiltonian

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\mu^2 x^2 + i\epsilon x^3 \quad (\epsilon \text{ real}) \qquad (2)$$

has a positive real spectrum [2, 3, 4]. Although this Hamiltonian is not Hermitian in the conventional sense, where Hermitian adjoint means complex conjugate and transpose, it still defines a unitary theory of quantum mechanics [5, 6]. This is because H in (2) is self-adjoint with respect to a new inner product that is distinct from the inner product of ordinary quantum mechanics [7].

For quantum-mechanical theories like that in (2) two issues need to be addressed. First, the spectrum of H must be shown to be real and positive. In Ref. [5] it was observed that spectral positivity of H is associated with unbroken space-time reflection symmetry ($\mathcal{PT}$ symmetry). [The term *unbroken* $\mathcal{PT}$ symmetry means that every eigenstate of H is also an eigenstate of $\mathcal{PT}$. This condition guarantees that the eigenvalues of H are real; H in (2) has an unbroken $\mathcal{PT}$ symmetry for all real ϵ.] Second, it is necessary to construct a Hilbert space with a positive norm on which the Hamiltonian acts and to show that H is self-adjoint with respect to this nonstandard inner product. Using a coordinate-space representation, this construction was made in Ref. [5] by introducing a new linear operator $\mathcal{C}$:

$$\mathcal{C}(x, y) \equiv \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y), \qquad (3)$$

where $\phi_n(x)$ $(n = 0, 1, 2, \ldots)$ are the eigenstates of H.

For the cubic Hamiltonian in (2), $\phi_n(x)$ satisfies the Schrödinger equation

$$-\tfrac{1}{2}\phi_n''(x) + \tfrac{1}{2}\mu^2 x^2 \phi_n(x) + i\epsilon x^3 \phi_n(x) = E_n \phi_n(x) \qquad (4)$$

and the boundary conditions $\phi_n(x) \to 0$ as $x \to \pm\infty$. The eigenstates of H are also eigenstates of $\mathcal{PT}$ and are normalized so that $\mathcal{PT}\phi_n(x) = \phi_n^*(-x) = \phi_n(x)$. The inner product involves $\mathcal{CPT}$ conjugation

$$\langle\psi|\chi\rangle_{\mathcal{CPT}} = \int dx\, \psi^{\mathcal{CPT}}(x)\chi(x),$$

where $\psi^{\mathcal{CPT}}(x) = \int dy\, \mathcal{C}(x, y)\psi^*(-y)$, instead of conventional Hermitian conjugation, where the inner product is $\langle\psi|\chi\rangle = \int dx\, \psi^*(x)\chi(x)$. The novelty of these complex Hamiltonians is that *the Hilbert space inner product is not prespecified; rather, it is dynamically determined by* H. This new kind of quantum theory is a sort of "bootstrap" theory because one must solve for the eigenstates of H before knowing what the Hilbert space and the associated inner product of the theory are.

The key problem in understanding a complex Hamiltonian like that in (1) or (2) is to determine the operator $\mathcal{C}$. In Ref. [8] perturbative methods were used to calculate $\mathcal{C}$ to third order in ϵ for the H in (2). The procedure was first to solve the Schrödinger equation (4) for $\phi_n(x)$ as a series in powers of ϵ. Then, $\phi_n(x)$ was substituted into (3) and the summation over n was performed to obtain $\mathcal{C}(x, y)$ to order ϵ^3. The result was complicated, but it simplified when $\mathcal{C}$ was rewritten in exponential form:

$$\mathcal{C}(x, y) = \left(e^{\epsilon Q_1 + \epsilon^3 Q_3 + \cdots}\right)\delta(x + y) + \mathcal{O}(\epsilon^5), \qquad (5)$$

where $Q_{2n+1}(x, p)$ are differential operators depending on x and $p = -i\frac{d}{dx}$. In the representation (5) only odd powers of ϵ appear in the exponent, the coefficients are real,

and the derivative operators act on the parity operator $\mathcal{P}=\delta(x+y)$. Also, $\mathcal{CP}=e^{\epsilon Q_1+\epsilon^3 Q_3+\cdots}$ is Hermitian.

Calculating $\mathcal{C}$ by direct evaluation of the sum in (3) is difficult in quantum mechanics because it is necessary to determine all the eigenfunctions of H. In quantum field theory such a procedure is impossible because there is no analog of the Schrödinger eigenvalue problem (4).

The breakthrough reported here is the discovery of a new and powerful method for calculating $\mathcal{C}$ by seeking an operator representation of it in the form $\mathcal{C}=e^{Q(x,p)}\mathcal{P}$. This representation, which is suggested by (5), has the advantage that $Q(x,p)$ is determined by elementary operator equations so that the eigenfunctions of H are not needed to find Q. Thus, the technique introduced here generalizes to quantum field theory.

We illustrate the generality of the representation $\mathcal{C}=e^{Q}\mathcal{P}$ by using two elementary examples:

Example 1: The complex 2×2 Hamiltonian

$$H=\begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta}\end{pmatrix}, \tag{6}$$

taken from Ref. [5], is $\mathcal{PT}$ symmetric, where $\mathcal{P}=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\mathcal{T}$ is complex conjugation. A nice way to express $\mathcal{C}$ in the region $s^2\geq r^2\sin^2\theta$ of unbroken $\mathcal{PT}$ symmetry is

$$\mathcal{C}=e^{Q}\mathcal{P},\quad Q=\tfrac{1}{2}\sigma_2\ln\left[(1-\sin\alpha)/(1+\sin\alpha)\right],$$

where $\sin\alpha=(r/s)\sin\theta$ and $\sigma_2=\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$ is the Pauli matrix. As $\theta\to 0$, H becomes Hermitian and $\mathcal{C}\to\mathcal{P}$.

Example 2: The Hamiltonian $H=\frac{1}{2}p^2+\frac{1}{2}x^2+i\epsilon x$ has an unbroken $\mathcal{PT}$ symmetry for all real ϵ. Its eigenvalues $E_n=n+\frac{1}{2}+\frac{1}{2}\epsilon$ are all real. The $\mathcal{C}$ operator is given exactly by $\mathcal{C}=e^{Q}\mathcal{P}$, where $Q=-\epsilon p$. Again, in the limit $\epsilon\to 0$ the Hamiltonian becomes Hermitian and $\mathcal{C}\to\mathcal{P}$.

Our new procedure for constructing $\mathcal{C}$ for a given H is based on three observations made in Ref. [5]: (i) $\mathcal{C}$ commutes with the space-time reflection operator $\mathcal{PT}$, but not with $\mathcal{P}$ or $\mathcal{T}$ separately; (ii) the square of $\mathcal{C}$ is the identity; (iii) $\mathcal{C}$ commutes with H. To summarize,

$$\text{(i) } [\mathcal{C},\mathcal{PT}]=0,\quad \text{(ii) } \mathcal{C}^2=\mathbf{1},\quad \text{(iii) } [\mathcal{C},H]=0. \tag{7}$$

Substituting $\mathcal{C}=e^{Q}\mathcal{P}$ into condition (i) in (7), we obtain $e^{Q(x,p)}=\mathcal{PT}e^{Q(x,p)}\mathcal{PT}=e^{Q(-x,p)}$, so $Q(x,p)$ is an *even* function of x. Next, we substitute $\mathcal{C}=e^{Q}\mathcal{P}$ into condition (ii) in (7) and get $e^{Q(x,p)}\mathcal{P}e^{Q(x,p)}\mathcal{P}=e^{Q(x,p)}e^{Q(-x,-p)}=1$, which implies that $Q(x,p)=-Q(-x,-p)$. Since $Q(x,p)$ is an even function of x, it must also be an *odd* function of p. Finally, substituting $\mathcal{C}=e^{Q(x,p)}\mathcal{P}$ into condition (iii) in (7), we obtain $e^{Q(x,p)}[\mathcal{P},H]+[e^{Q(x,p)},H]\mathcal{P}=0$. All of the Hamiltonians H considered in this paper have the form $H=H_0+\epsilon H_1$, where H_0 is a free Hamiltonian that commutes with the parity operator $\mathcal{P}$ and H_1 represents a cubic interaction term, which *anticommutes* with $\mathcal{P}$. Hence, for quantum theories described by cubic Hamiltonians, we have

$$\epsilon e^{Q(x,p)}H_1=[e^{Q(x,p)},H]. \tag{8}$$

For all cubic Hamiltonians, $Q(x,p)$ may be expanded as a series in *odd* powers of ϵ as in (5): $Q(x,p)=\epsilon Q_1(x,p)+\epsilon^3 Q_3(x,p)+\epsilon^5 Q_5(x,p)+\cdots$. (In quantum field theory we can interpret the coefficients Q_{2n+1} as interaction vertices of $2n+3$ factors of the quantum fields.) Substituting this expansion into (8) and collecting the coefficients of like powers of ϵ, we obtain a sequence of operator equations that can be solved systematically for the operator-valued functions $Q_n(x,p)$ $(n=1,3,5,\ldots)$ subject to the constraints that $Q(x,p)$ be even in x and odd in p. It is interesting that even powers of ϵ yield redundant information because the equations arising from the coefficient of ϵ^{2n} can be derived from the equations arising from the coefficients of ϵ^{2n-1}, ϵ^{2n-3}, …, ϵ. The first three operator equations for Q_n are

$$\begin{aligned}
[H_0,Q_1] &= -2H_1,\\
[H_0,Q_3] &= -\tfrac{1}{6}[Q_1,[Q_1,H_1]],\\
[H_0,Q_5] &= \tfrac{1}{360}[Q_1,[Q_1,[Q_1,[Q_1,H_1]]]]\\
&\quad -\tfrac{1}{6}\Big([Q_1,[Q_3,H_1]]+[Q_3,[Q_1,H_1]]\Big).
\end{aligned} \tag{9}$$

For H in (2) we solve the equations in (9) by substituting the most general polynomial form for Q_n using arbitrary coefficients and then solving for these coefficients. For example, to solve the first equation in (9), $[H_0,Q_1]=-2ix^3$, we take as an *ansatz* for Q_1 the most general Hermitian cubic polynomial that is even in x and odd in p: $Q_1(x,p)=Mp^3+Nxpx$, where M and N are unknown coefficients. The operator equation for Q_1 is satisfied if $M=-\frac{4}{3}\mu^{-4}$ and $N=-2\mu^{-2}$.

To present the solutions for $Q_n(x,p)$ we use $S_{m,n}$ to represent the *totally symmetrized* sum over terms containing m factors of p and n factors of x [9]. Thus, $S_{0,0}=1$, $S_{0,3}=x^3$, $S_{1,1}=\frac{1}{2}(xp+px)$, $S_{1,2}=\frac{1}{3}\left(x^2p+xpx+px^2\right)$, and so on. We have solved (9) for Q_1, Q_3, Q_5, and Q_7 in closed form in terms of $S_{m,n}$:

$$\begin{aligned}
Q_1 &= -\tfrac{4}{3}\mu^{-4}p^3-2\mu^{-2}S_{1,2},\\
Q_3 &= \tfrac{128}{15}\mu^{-10}p^5+\tfrac{40}{3}\mu^{-8}S_{3,2}+8\mu^{-6}S_{1,4}-12\mu^{-8}p,\\
Q_5 &= -\tfrac{320}{3}\mu^{-16}p^7-\tfrac{544}{3}\mu^{-14}S_{5,2}-\tfrac{512}{3}\mu^{-12}S_{3,4}\\
&\quad -64\mu^{-10}S_{1,6}+\tfrac{24\,736}{45}\mu^{-14}p^3+\tfrac{6\,368}{15}\mu^{-12}S_{1,2},\\
Q_7 &= \tfrac{553\,984}{315}\mu^{-22}p^9+\tfrac{97\,792}{35}\mu^{-20}S_{7,2}+\tfrac{377\,344}{105}\mu^{-18}S_{5,4}\\
&\quad +\tfrac{721\,024}{315}\mu^{-16}S_{3,6}+\tfrac{1\,792}{3}\mu^{-14}S_{1,8}-\tfrac{2\,209\,024}{105}\mu^{-20}p^5\\
&\quad -\tfrac{2\,875\,648}{105}\mu^{-18}S_{3,2}-\tfrac{390\,336}{35}\mu^{-16}S_{1,4}+\tfrac{46\,976}{5}\mu^{-18}p.
\end{aligned} \tag{10}$$

These results constitute a seventh-order perturbative expansion of $\mathcal{C}$ in terms of the operators x and p.

The representation $\mathcal{C}=e^{Q}\mathcal{P}$ bears a strong resemblance to the WKB *ansatz*, which is also an exponential of a power series. Like our calculation of $\mathcal{C}$ here, WKB methods determine the energy eigenvalues E_n *to all orders* in powers of $\hbar$ via a system of equations like those in (9) without ever using the eigenfunctions $\phi_n(x)$ [10]. Moreover, only the *even* terms in the WKB series are

needed to find E_n; the odd terms in the series are redundant and provide no information about E_n [10]. The difference between a conventional WKB series and the series representation for $Q(x,p)$ is that the first term in a WKB series is proportional to $\hbar^{-1}$ while the expansion for Q contains only positive powers of ϵ. However, based on the nonperturbative calculations in Ref. [8], we believe that for a $\mathcal{PT}$-symmetric $-\epsilon x^4$ theory, the first term in the expansion of Q will also be proportional to ϵ^{-1}.

The massless (strong-coupling) limit $\mu \to 0$ of $\mathcal{C}$ for H in (1) is interesting because it is singular. [The dimensionless perturbation parameter is $\epsilon\mu^{-5/2}$, so negative powers of μ appear in every order in (10). Thus, as $\mu \to 0$, the perturbation series for Q ceases to exist.] As $\mu \to 0$, a new *nonpolynomial* representation for Q emerges. To find $\mathcal{C}$ when $\mu = 0$ we return to the operator equations in (9) and seek new solutions for the special case $H_0 = \frac{1}{2}p^2$. The situation here is like that in Ref. [9], where the objective was to calculate the time operator in quantum mechanics. Here, as in the case of the time operator, we use Weyl ordering to generalize $S_{m,n}$ from positive m to negative m. For example, $S_{-1,1} = \frac{1}{2}\left(x\frac{1}{p} + \frac{1}{p}x\right)$. The solution to the first equation in (9), $\left[\frac{1}{2}p^2, Q_1\right] = -2ix^3$, is $Q_1 = \frac{1}{2}S_{-1,4} + \alpha S_{-5,0}$, where α is arbitrary. This solution is odd in p, even in x, and has the same dimensions as Q_1 in (10). Also, $Q_3 = \left(\frac{3}{32} + \frac{305}{8}\alpha\right) S_{-11,4} - \left(\frac{135}{16} + \frac{5773}{8}\alpha - \frac{75}{12}\alpha^2\right) S_{-13,2} + \left(\frac{7}{16} + 20\alpha\right) S_{-9,6} - \frac{3}{32}S_{-7,8} + \frac{1}{40}S_{-5,10} + \beta S_{-15,0}$, where β is a new arbitrary number.

The new operator techniques introduced in this paper extend to systems having several dynamical degrees of freedom. We have calculated $\mathcal{C}$ to third order in ϵ for

$$H = \tfrac{1}{2}\left(p_x^2 + p_y^2\right) + \tfrac{1}{2}\left(x^2 + y^2\right) + i\epsilon x^2 y, \tag{11}$$

which has two degrees of freedom. The result is

$$\begin{aligned} Q_1(x,y,p,q) &= -\tfrac{4}{3}p^2 q - \tfrac{1}{3}S_{1,1}y - \tfrac{2}{3}x^2 q, \\ Q_3(x,y,p,q) &= \tfrac{512}{405}p^2q^3 + \tfrac{512}{405}p^4 q + \tfrac{1088}{405}S_{1,1}T_{2,1} \\ &-\tfrac{256}{405}p^2T_{1,2} + \tfrac{512}{405}S_{3,1}y + \tfrac{288}{405}S_{2,2}q - \tfrac{32}{405}x^2q^3 \\ &+\tfrac{736}{405}x^2T_{1,2} - \tfrac{256}{405}S_{1,1}y^3 + \tfrac{608}{405}S_{1,3}y - \tfrac{128}{405}x^4 q - \tfrac{8}{9}q, \end{aligned} \tag{12}$$

where $T_{m,n}$ represents a totally symmetric product of m factors of q and n factors of y. For the Hamiltonian

$$H = \tfrac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + \tfrac{1}{2}\left(x^2 + y^2 + z^2\right) + i\epsilon xyz, \tag{13}$$

which has three degrees of freedom, we have

$$\begin{aligned} Q_1(x,y,z,p,q,r) &= -\tfrac{2}{3}(yzp + xzq + xyr) - \tfrac{4}{3}pqr, \\ Q_3(x,y,z,p,q,r) &= \tfrac{128}{405}\left(p^3qr + q^3pr + r^3qp\right) \\ &+\tfrac{136}{405}[pxp(yr+zq) + qyq(xr+zp) + rzr(xq+yp)] \\ &-\tfrac{64}{405}(xpxqr + yqypr + zrzpq) + \tfrac{184}{405}(xpxyz + yqyxz \\ &+zrzxy) - \tfrac{32}{405}\left[x^3(yr+zq) + y^3(xr+zp)\right. \\ &\left.+z^3(xq+yp)\right] - \tfrac{8}{405}\left(p^3yz + q^3xz + r^3xy\right). \end{aligned} \tag{14}$$

Note that the technique used in Ref. [11] to calculate $\mathcal{C}$ gives Q_1 in (12) and (14), but it becomes hopelessly difficult beyond first order because one encounters the problems associated with degenerate energy levels. Our new method works to all orders of perturbation theory because the eigenfunctions are not needed to obtain $\mathcal{C}$.

We now show how to apply the powerful techniques we have developed in quantum mechanics to the calculation of $\mathcal{C}$ in quantum field theory. As before, we express $\mathcal{C}$ in the form $\mathcal{C} = e^{\epsilon Q_1 + \epsilon^3 Q_3 + \cdots}\mathcal{P}$, where now Q_{2n+1} ($n = 0, 1, 2, \ldots$) are real functionals of the fields $\varphi_{\mathbf{x}}$ and $\pi_{\mathbf{x}}$. To find Q_n for H in (1) we must again solve the operator equations in (9). In terms of the inverse Green's function $G^{-1}_{\mathbf{xy}} \equiv (\mu^2 - \nabla^2_{\mathbf{x}})\delta(\mathbf{x}-\mathbf{y})$ the first equation in (9) is

$$\left[\int d\mathbf{x}\,\pi_{\mathbf{x}}^2 + \iint d\mathbf{x}\,d\mathbf{y}\,\varphi_{\mathbf{x}}G^{-1}_{\mathbf{xy}}\varphi_{\mathbf{y}}, Q_1\right] = -4i\int d\mathbf{x}\,\varphi_{\mathbf{x}}^3. \tag{15}$$

Recalling the result for Q_1 in (10) we try the *ansatz*

$$Q_1 = \iiint d\mathbf{x}\,d\mathbf{y}\,d\mathbf{z}\left(M_{(\mathbf{xyz})}\pi_{\mathbf{x}}\pi_{\mathbf{y}}\pi_{\mathbf{z}} + N_{\mathbf{x}(\mathbf{yz})}\varphi_{\mathbf{y}}\pi_{\mathbf{x}}\varphi_{\mathbf{z}}\right).$$

The notation $M_{(\mathbf{xyz})}$ indicates that this function is totally symmetric in its three arguments and the notation $N_{\mathbf{x}(\mathbf{yz})}$ indicates that this function is symmetric under the interchange of the second and third arguments. The unknown functions M and N are form factors; they describe the spatial distribution of three-point interactions of the field variables in Q_1. *The nonlocal spatial interaction of the fields is an intrinsic property of* $\mathcal{C}$.

To determine M and N we substitute Q_1 into (15) and obtain a coupled system of differential equations:

$$\begin{aligned} (\mu^2 - \nabla^2_{\mathbf{x}})N_{\mathbf{x}(\mathbf{yz})} + (\mu^2 - \nabla^2_{\mathbf{y}})N_{\mathbf{y}(\mathbf{xz})} + (\mu^2 - \nabla^2_{\mathbf{z}})N_{\mathbf{z}(\mathbf{xy})} \\ = -6\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{x}-\mathbf{z}), \\ N_{\mathbf{x}(\mathbf{yz})} + N_{\mathbf{y}(\mathbf{xz})} = 3(\mu^2 - \nabla^2_{\mathbf{z}})M_{(\mathbf{wxy})}. \end{aligned} \tag{16}$$

We solve these differential equations by Fourier transforming to momentum space, to obtain

$$M_{(\mathbf{xyz})} = -\frac{4}{(2\pi)^{2D}}\iint d\mathbf{p}\,d\mathbf{q}\frac{e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{p} + i(\mathbf{x}-\mathbf{z})\cdot\mathbf{q}}}{D(\mathbf{p},\mathbf{q})}, \tag{17}$$

where $D(\mathbf{p},\mathbf{q}) = 4[\mathbf{p}^2\mathbf{q}^2 - (\mathbf{p}\cdot\mathbf{q})^2] + 4\mu^2(\mathbf{p}^2 + \mathbf{p}\cdot\mathbf{q} + \mathbf{q}^2) + 3\mu^4$ is positive, and

$$N_{\mathbf{x}(\mathbf{yz})} = 3\left(\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}} + \tfrac{1}{2}\mu^2\right)M_{(\mathbf{xyz})}. \tag{18}$$

For the special case of a $(1+1)$-dimensional quantum field theory the integral in (17) evaluates to

$$M_{(\mathbf{xyz})} = -\left(\sqrt{3}\pi\mu^2\right)^{-1}\mathrm{K}_0(\mu R), \tag{19}$$

where K_0 is the associated Bessel function and $R^2 = \frac{1}{2}[(\mathbf{x}-\mathbf{y})^2 + (\mathbf{y}-\mathbf{z})^2 + (\mathbf{z}-\mathbf{x})^2]$.

We can perform the same calculations for cubic quantum field theories having several interacting scalar fields.

Consider first the case of *two* scalar fields $\varphi_{\mathbf{x}}^{(1)}$ and $\varphi_{\mathbf{x}}^{(2)}$ whose interaction is governed by $H = H_0^{(1)} + H_0^{(2)} + i\epsilon \int d\mathbf{x} \left[\varphi_{\mathbf{x}}^{(1)}\right]^2 \varphi_{\mathbf{x}}^{(2)}$, which is the analog of the quantum-mechanical theory described by H in (11). Here,

$$H_0^{(j)} = \tfrac{1}{2}\int d\mathbf{x} \left[\pi_{\mathbf{x}}^{(j)}\right]^2 + \tfrac{1}{2}\iint d\mathbf{x}\, d\mathbf{y} \left[G_{\mathbf{xy}}^{(j)}\right]^{-1} \varphi_{\mathbf{x}}^{(j)} \varphi_{\mathbf{y}}^{(j)}.$$

To determine $\mathcal{C}$ to order ϵ we introduce the *ansatz*

$$\begin{aligned} Q_1 \;=& \iiint d\mathbf{x}\, d\mathbf{y}\, d\mathbf{z} \Big[N_{\mathbf{xyz}}^{(1)} \left(\pi_{\mathbf{z}}^{(1)} \varphi_{\mathbf{y}}^{(1)} + \varphi_{\mathbf{y}}^{(1)} \pi_{\mathbf{z}}^{(1)}\right) \varphi_{\mathbf{x}}^{(2)} \\ &+ N_{\mathbf{x(yz)}}^{(2)} \pi_{\mathbf{x}}^{(2)} \varphi_{\mathbf{y}}^{(1)} \varphi_{\mathbf{z}}^{(1)} + M_{\mathbf{x(yz)}} \pi_{\mathbf{x}}^{(2)} \pi_{\mathbf{y}}^{(1)} \pi_{\mathbf{z}}^{(1)} \Big], \end{aligned}$$

where $M_{\mathbf{x(yz)}}$, $N_{\mathbf{xyz}}^{(1)}$, and $N_{\mathbf{x(yz)}}^{(2)}$ are unknown functions and the parentheses indicate symmetrization. We get

$$M_{\mathbf{x(yz)}} = -\frac{4}{(2\pi)^{2D}} \iint d\mathbf{q}\, d\mathbf{r}\, \frac{e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{q} + i(\mathbf{x}-\mathbf{z})\cdot\mathbf{r}}}{\mathcal{D}(\mathbf{q},\mathbf{r})}, \quad (20)$$

where $\mathcal{D}(\mathbf{q},\mathbf{r}) = 4[\mathbf{q}^2\mathbf{r}^2 - (\mathbf{q}\cdot\mathbf{r})^2] + 4\mu_1^2(\mathbf{q}+\mathbf{r})^2 - 4\mu_2^2\mathbf{q}\cdot\mathbf{r} - \mu_2^4 + 4\mu_1^2\mu_2^2$, and

$$\begin{aligned} N_{\mathbf{xyz}}^{(1)} &= \left[-\nabla_{\mathbf{y}}^2 - \nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}} + \tfrac{1}{2}\mu_2^2\right] M_{\mathbf{x(yz)}}, \\ N_{\mathbf{x(yz)}}^{(2)} &= \left[\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}} - \mu_1^2 - \tfrac{1}{2}\mu_2^2\right] M_{\mathbf{x(yz)}}. \end{aligned} \quad (21)$$

For *three* interacting scalar fields whose dynamics is described by $H = H_0^{(1)} + H_0^{(2)} + H_0^{(3)} + i\epsilon \int d\mathbf{x}\, \varphi_{\mathbf{x}}^{(1)} \varphi_{\mathbf{x}}^{(2)} \varphi_{\mathbf{x}}^{(3)}$, which is the analog of H in (13), we make the *ansatz*

$$\begin{aligned} Q_1 \;=& \iiint d\mathbf{x}\, d\mathbf{y}\, d\mathbf{z} \Big[N_{\mathbf{xyz}}^{(1)} \pi_{\mathbf{x}}^{(1)} \varphi_{\mathbf{y}}^{(2)} \varphi_{\mathbf{z}}^{(3)} + N_{\mathbf{xyz}}^{(2)} \pi_{\mathbf{x}}^{(2)} \varphi_{\mathbf{y}}^{(3)} \varphi_{\mathbf{z}}^{(1)} \\ &+ N_{\mathbf{xyz}}^{(3)} \pi_{\mathbf{x}}^{(3)} \varphi_{\mathbf{y}}^{(1)} \varphi_{\mathbf{z}}^{(2)} + M_{\mathbf{xyz}} \pi_{\mathbf{x}}^{(1)} \pi_{\mathbf{y}}^{(2)} \pi_{\mathbf{z}}^{(3)} \Big]. \end{aligned}$$

The solutions for the unknown functions are as follows: $M_{\mathbf{xyz}}$ is given by the integral (17) with the more general formula $D(\mathbf{p},\mathbf{q}) = 4[\mathbf{p}^2\mathbf{q}^2 - (\mathbf{p}\cdot\mathbf{q})^2] + 4[\mu_1^2(\mathbf{q}^2 + \mathbf{p}\cdot\mathbf{q}) + \mu_2^2(\mathbf{p}^2 + \mathbf{p}\cdot\mathbf{q}) - \mu_3^2\mathbf{p}\cdot\mathbf{q}] + \mu^4$ with $\mu^4 = 2\mu_1^2\mu_2^2 + 2\mu_1^2\mu_3^2 + 2\mu_2^2\mu_3^2 - \mu_1^4 - \mu_2^4 - \mu_3^4$. The N coefficients are expressed as derivatives acting on M:

$$\begin{aligned} N_{\mathbf{xyz}}^{(1)} &= \left[4\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}} + 2(\mu_2^2 + \mu_3^2 - \mu_1^2)\right] M_{\mathbf{xyz}}, \\ N_{\mathbf{xyz}}^{(2)} &= \left[-4\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}} - 4\nabla_{\mathbf{z}}^2 + 2(\mu_1^2 + \mu_3^2 - \mu_2^2)\right] M_{\mathbf{xyz}}, \\ N_{\mathbf{xyz}}^{(3)} &= \left[-4\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}} - 4\nabla_{\mathbf{y}}^2 + 2(\mu_1^2 + \mu_2^2 - \mu_3^2)\right] M_{\mathbf{xyz}}. \end{aligned}$$

Our new perturbative calculation of $\mathcal{C}$ for cubic quantum field theories is an important step in our ongoing program to obtain new physical models by extending quantum mechanics and quantum field theory into the complex domain. The operator $\mathcal{C}$ is a new conserved quantity in quantum field theory and this operator is required to construct observables and to evaluate matrix elements of field operators.

We hope to generalize the breakthrough reported in this paper to noncubic $\mathcal{PT}$-symmetric quantum field theories such as a (3+1)-dimensional $-g\varphi^4$ field theory. This remarkable model field theory has a positive spectrum, is renormalizable, is asymptotically free [12], and has a nonzero one-point Green's function $G_1 = \langle\varphi\rangle$. Consequently, this model may ultimately help to elucidate the dynamics of the Higgs sector of the standard model.

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*Permanent address: Department of Physics, Washington University, St. Louis, MO 63130, USA.

[1] R. Brower, M. Furman, and M. Moshe, Phys. Lett. B **76**, 213 (1978).

[2] E. Caliceti, S. Graffi, and M. Maioli, Comm. Math. Phys. **75**, 51 (1980).

[3] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).

[4] P. Dorey, C. Dunning and R. Tateo, J. Phys. A **34** L391 (2001); *ibid.* **34**, 5679 (2001).

[5] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270402 (2002) and Am. J. Phys. **71**, 1095 (2003).

[6] A. Mostafazadeh, J. Math. Phys. **43**, 3944 (2002).

[7] In M. Reed and B. Simon, *Functional Analysis I* (Academic, New York, 1980), a unitary map U is defined as an isomorphism between Hilbert spaces (p. 39). This broad definition led to the false claim that quantum theories having distinct Hermitian inner products are related by unitary transformations satisfying $U^\dagger U = \mathbf{1}$ [A. Mostafazadeh, J. Phys. A: Math. Gen. **36**, 7081 (2003)]. The correct statement is: Let $\mathcal{H}_1$ be the Hilbert space of ordinary quantum mechanics and $\mathcal{H}_2$ be that of a $\mathcal{PT}$-symmetric quantum theory. Then the isomorphism $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ satisfies $U^\dagger U = \mathcal{CP} = e^Q \neq \mathbf{1}$.

[8] C. M. Bender, P. N. Meisinger, and Q. Wang, J. Phys. A **36**, 1973 (2003).

[9] C. M. Bender and G. V. Dunne, Phys. Rev. D **40**, 2739 and 3504 (1989).

[10] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, (McGraw-Hill, New York, 1978), Chap. 10.

[11] C. M. Bender, J. Brod, A. T. Refig, and M. E. Reuter, Imperial College Preprint (2003).

[12] C. M. Bender, K. A. Milton, and V. M. Savage, Phys. Rev. D **62**, 85001 (2000).